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Positive solutions of second-order delay differential equations with a damping term

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ABSTRACT

In the paper, the existence of positive solutions is studied for the second-order delay differential equation with a damping term

$$\ddot{x}(t)+a(t)\dot{x}(t)+b(t)x(h(t))=0$$

using a comparison with the integro-differential equation

$$\dot{y}(t)+\int_{t_0}^{t} e^{-\int_s^t a(\xi)d\xi} b(s)y(h(s))ds=0.$$

Explicit non-oscillation criteria and comparison type results are derived.

1. Introduction

In this paper, we consider a second-order linear scalar differential equation

$$\ddot{x}(t)+a(t)\dot{x}(t)+b(t)x(h(t))=f(t), \tag{1}$$

and the corresponding homogeneous equation

$$\ddot{x}(t)+a(t)\dot{x}(t)+b(t)x(h(t))=0. \tag{2}$$

Such linear and related nonlinear equations attract the attention of many mathematicians because of their significance in applications. Here, we just mention the monographs by Myshkis [1], Norkin [2], Ladde et al. [3], Györi and Ladas [4], Erbe et al. [5], Burton [6,7], Kolmanovsky and Nosov [8], Agarwal et al. [9] and references therein.

In particular, Minorsky [10] in 1962 considered the problem of stabilizing the rolling of a ship by the "activated tanks method" in which ballast water is pumped from one position to another. To solve this problem, he constructed several delay differential equations with damping of the form (1) and (2).

In spite of the obvious importance in applications, there are only a few papers on delay differential equations with damping.

In [11] the authors considered the autonomous equation (2) and obtained stability results by analysing the roots of the characteristic equation.

In [6] the stability of the autonomous equation

$$\ddot{x}(t)+a\dot{x}(t)+bx(t-\tau)=0 \tag{3}$$

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was studied using Lyapunov functions. It was demonstrated that, if $a > 0$, $b > 0$ and $b\tau < a$, then Eq. (3) is exponentially stable. Other results obtained by the method of Lyapunov functions can be found in [12,13].

In a recent paper [14], stability properties for the following linear delay equation

$$\ddot{x}(t) + a(t)\dot{x}(g(t)) + b(t)x(h(t)) = 0 \tag{4}$$

are considered.

In the present paper, we study the nonoscillation problem for Eq. (1). There are only a few papers dealing with such problems.

To nonoscillation, we apply some ideas from [15], where the nonoscillation properties of a second-order ordinary differential equation are compared with similar properties of a first-order integro-differential equation.

We compare the nonoscillation properties of a delayed differential equation (2) with an integro-differential equation with delay. To the best of our knowledge, such integro-differential equations with delay are considered here for the first time.

Other approaches to the nonoscillation of second-order equations with a damping term were proposed in [16]. Our results differ from the results obtained in that paper.

2. Preliminaries

Throughout this paper, we assume (unless no more restrictive conditions are assumed) that:

(a1) $a, b, f: [0, \infty) \to \mathbb{R}$ are Lebesgue measurable and essentially bounded functions;

(a2) $h: [0, \infty) \to \mathbb{R}$ is a Lebesgue measurable function, $h(t) \le t, t \ge 0, \limsup_{t\to\infty}(t - h(t)) < \infty$.

In addition to (1), consider for each $t_0 \ge 0$ an initial value problem

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(h(t)) = f(t), \quad t \ge t_0, \tag{5}$$

$$x(t) = \varphi(t), \quad t < t_0, \qquad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0 \tag{6}$$

and, throughout this paper, we also assume that the following hypothesis holds:

(a3) $\varphi: (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

Definition 1. A function $x: \mathbb{R} \to \mathbb{R}$ with derivative $\dot{x}$, locally absolutely continuous on $[t_0, \infty)$, is called a solution of problem (5), (6) if it satisfies Eq. (5) for almost every $t \in [t_0, \infty)$ and equalities (6) for $t \le t_0$.

Definition 2. For each $s \ge 0$, the solution $X(t, s)$ of the problem

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(h(t)) = 0, \quad t \ge s,$$
$$x(t) = 0, \quad t \le s, \qquad \dot{x}(s) = 1$$

is called the fundamental function of Eq. (2).

We underline that, obviously, $X(t, s) = 0,\ t \le s$.

Let functions x_1 and x_2 be the solutions of the following problem:

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(h(t)) = 0, \quad t \ge t_0,$$
$$x(t) = 0, \quad t < t_0$$

with the initial values

$$x_1(t_0) = 1, \qquad \dot{x}_1(t_0) = 0$$

for x_1 and

$$x_2(t_0) = 0, \qquad \dot{x}_2(t_0) = 1$$

for x_2. Such functions x_1 and x_2 are called *the fundamental system of solutions* of Eq. (2) (in short "*the fundamental system*"). It is easy to see that $x_2(t) = X(t, t_0)$.

Lemma 1 (*[17]*)**.** *Let* (a1)–(a3) *be true. Then there exists exactly one solution of problem* (5), (6) *that can be represented in the form*

$$x(t) = x_1(t)x_0 + x_2(t)\dot{x}_0 + \int_{t_0}^{t} X(t, s)f(s)\mathrm{d}s - \int_{t_0}^{t} X(t, s)b(s)\varphi(h(s))\mathrm{d}s, \quad t \ge t_0$$

where we set $\varphi(h(s)) := 0$ *if* $h(s) \ge t_0$.

3. Integro–differential equation

To obtain positiveness conditions for the fundamental function of Eq. (2) in this section, we consider a similar problem for the following *integro-differential equation with delay*

$$\dot{y}(t)+\int_{t_0}^{t} e^{-\int_s^t a(\xi)d\xi} b(s)y(h(s))ds=0, \quad t\geq t_0\geq 0. \tag{7}$$

We assume that, for the functions a, b and h, conditions (a1) and (a2) hold.

In addition to (7), we consider, for each $t_0\geq 0$, the initial value problem

$$\dot{y}(t)+\int_{t_0}^{t} e^{-\int_s^t a(\xi)d\xi} b(s)y(h(s))ds=f(t), \quad t\geq t_0, \tag{8}$$

$$y(t)=\varphi(t), \quad t<t_0, \qquad y(t_0)=y_0. \tag{9}$$

We assume that conditions (a1), (a3) hold for the functions f, φ.

Definition 3. A function $y:\mathbb{R}\to\mathbb{R}$, locally absolutely continuous on $[t_0,\infty)$, is called a solution of problem (8), (9) if it satisfies Eq. (8) for almost every $t\in[t_0,\infty)$ and equalities (9) for $t\leq t_0$.

If a differential inequality, rather than an equation, is considered (see, e.g., inequality (11) below), its solution y, such that $y:\mathbb{R}\to\mathbb{R}$, is defined in much the same way.

Definition 4. For each $s\geq 0$, the solution $Y(t,s)$ of the problem

$$\dot{y}(t)+\int_{s}^{t} e^{-\int_\tau^t a(\xi)d\xi} b(\tau)y(h(\tau))d\tau=0, \quad t\geq s,$$
$$y(t)=0, \quad t<s, \qquad y(s)=1,$$

is called the fundamental function of Eq. (7).

Obviously, the property $Y(t,s)=0$, $t<s$ holds.

Lemma 2 ([18]). *Let (a1)–(a3) hold. Then there exists one and only one solution of problem (8), (9) that can be represented in the form*

$$y(t)=Y(t,t_0)y_0+\int_{t_0}^{t} Y(t,s)f(s)ds-\int_{t_0}^{t} Y(t,s)b(s)\psi(s)ds, \quad t\geq t_0 \tag{10}$$

where

$$\psi(s)=\int_{t_0}^{s} e^{-\int_q^s a(\xi)d\xi} b(q)\varphi(h(q))dq$$

and $\varphi(h(q)):=0$ *if* $h(q)\geq t_0$.

Let us find conditions under which Eq. (7) has a positive solution.

In what follows, we will apply the following result [19, Theorem 2.1, page 18] (see also [20, Theorem B.1, page 284]). Let $L[t_0,T]$, $t_0<T$ be the space of all functions integrable on $[t_0,T]$ with the norm $\|f\|=\int_{t_0}^{T}|f(s)|ds$.

Lemma 3. *Let the function $r(t,s)$ be measurable on the square $[a,b]\times[a,b]$, let the function $r(t,s)$, for almost every $t\in[a,b]$, have finite one-sided limits at each point $s\in[a,b]$, and let there exist a function $v\in L[t_0,T]$, $a\leq t_0<T\leq b$ such that $|r(t,s)|\leq v(t)$ for each $(t,s)\in[t_0,T]\times[a,b]$. Then the integral operator*

$$(Ky)(t)=\int_a^b r(t,s)y(s)ds$$

maps $L[t_0,T]$ to $L[t_0,T]$ and is a compact operator in this space.

Theorem 1. *Let $a,b:[0,\infty)\to[0,\infty)$. Then the following conditions are equivalent.*

(1) There exists a $t_1\geq 0$ and a solution $y:\mathbb{R}\to\mathbb{R}$ of the problem

$$\dot{y}(t)+\int_{t_1}^{t} e^{-\int_s^t a(\xi)d\xi} b(s)y(h(s))ds\leq 0, \quad t\geq t_1, \tag{11}$$
$$y(t)=0, \quad t<t_1$$

which is positive for $t\geq t_1$.

(2) For some $t_2 \geq 0$, there exists a locally essentially bounded function $u: \mathbb{R} \to \mathbb{R}$ such that

$$u(t) \geq \int_{t_2}^{t} e^{-\int_s^t a(\xi)d\xi} e^{\int_{\max\{t_2,h(s)\}}^t u(\xi)d\xi} b(s)ds, \quad t \geq t_2. \tag{12}$$

(3) For some $t_3 \geq 0$, we have: $Y(t,s) > 0$, $t > s \geq t_3$.

(4) For some $t_4 \geq 0$, there exists a solution $y(t)$ of problem (7), (9) (with t_0 changed to t_4) positive on $[t_4, \infty)$ and with $\varphi(t) \equiv 0$ for $t < t_4$.

Proof. (1) $\Longrightarrow$ (2). Let $y(t) > 0$, $t \geq t_1$ be a solution of (11). Hence, on any bounded subinterval of $[t_1, \infty)$, we have $y(t) \geq \alpha > 0$ where α depends on the subinterval chosen. Then we can define

$$u(t) = \begin{cases} -\dfrac{\dot{y}(t)}{y(t)} \geq 0 & \text{for almost every } t \geq t_1, \\ 0, \quad t < t_1, \end{cases}$$

which is an essentially locally bounded function. We also have

$$y(t) = \begin{cases} y(t_1)e^{-\int_{t_1}^t u(\xi)d\xi} & \text{for almost every } t \geq t_1, \\ 0, \quad t < t_1, \end{cases}$$

$$\dot{y}(t) = \begin{cases} -y(t_1)u(t)e^{-\int_{t_1}^t u(\xi)d\xi} & \text{for almost every } t \geq t_1, \\ 0, \quad t < t_1. \end{cases}$$

Hence

$$y(h(t)) = y(t_1)e^{-\int_{t_1}^{\max\{t_1,h(t)\}} u(\xi)d\xi} \chi_1(h(t)),$$

where

$$\chi_1(s) = \begin{cases} 1 & \text{if } s \geq t_1, \\ 0 & \text{if } s < t_1. \end{cases}$$

Substituting $y, \dot{y}$ into the left-hand side of inequality (11), we have

$$\begin{aligned}
\dot{y}(t) + \int_{t_1}^{t} e^{-\int_s^t a(\xi)d\xi} b(s)y(h(s))ds &= -y(t_1)u(t)e^{-\int_{t_1}^t u(\xi)d\xi} \\
&\quad + y(t_1)\int_{t_1}^{t} e^{-\int_s^t a(\xi)d\xi} \chi_1(h(s))b(s)e^{-\int_{t_1}^{\max\{t_1,h(s)\}} u(\xi)d\xi} ds \\
&= -y(t_1)e^{-\int_{t_1}^t u(\xi)d\xi} \left[u(t) - \int_{t_1}^{t} e^{-\int_s^t a(\xi)d\xi} e^{\int_{\max\{t_1,h(s)\}}^t u(\xi)d\xi} \chi_1(h(s))b(s)ds\right] \\
&\leq 0.
\end{aligned}$$

Hence, $u(t)$ satisfies

$$u(t) \geq \int_{t_1}^{t} e^{-\int_s^t a(\xi)d\xi} e^{\int_{\max\{t_1,h(s)\}}^t u(\xi)d\xi} \chi_1(h(s))b(s)ds.$$

By (a2), there exists $t_2 \geq t_1$ such that $h(t) \geq t_1$ for $t \geq t_2$. Then, for $t \geq t_2$, we have

$$\begin{aligned}
u(t) &\geq \int_{t_1}^{t_2} e^{-\int_s^t a(\xi)d\xi} e^{\int_{\max\{t_1,h(s)\}}^t u(\xi)d\xi} \chi_1(h(s))b(s)ds + \int_{t_2}^{t} e^{-\int_s^t a(\xi)d\xi} e^{\int_{\max\{t_1,h(s)\}}^t u(\xi)d\xi} b(s)ds \\
&\geq \int_{t_2}^{t} e^{-\int_s^t a(\xi)d\xi} e^{\int_{\max\{t_2,h(s)\}}^t u(\xi)d\xi} b(s)ds \geq 0.
\end{aligned}$$

(2) $\Longrightarrow$ (3). Let $y(t)$ be the solution of the initial value problem

$$\begin{aligned}
&\dot{y}(t) + \int_{t_2}^{t} e^{-\int_s^t a(\xi)d\xi} b(s)y(h(s))ds = f(t), \quad t \geq t_2, \\
&y(t) = 0, \quad t \leq t_2.
\end{aligned} \tag{13}$$

Recall that the existence and unicity of a solution is a consequence of Lemma 2.

Denote $z(t) = \dot{y}(t) + u(t)y(t)$ where $u(t)$ is a nonnegative solution of inequality (12). Then

$$y(t) = \begin{cases} \int_{t_2}^{t} \mathrm{e}^{-\int_s^t u(\xi)\mathrm{d}\xi} z(s)\mathrm{d}s, & t \geq t_2, \\ 0, & t < t_2, \end{cases} \tag{14}$$

$$\dot{y}(t) = \begin{cases} z(t) - u(t)\int_{t_2}^{t} \mathrm{e}^{-\int_s^t u(\xi)\mathrm{d}\xi} z(s)\mathrm{d}s, & t \geq t_2, \\ 0, & t < t_2. \end{cases} \tag{15}$$

Hence

$$y(h(t)) = \left(\int_{t_2}^{\max\{t_2,h(t)\}} \mathrm{e}^{-\int_s^{\max\{t_2,h(t)\}} u(\xi)\mathrm{d}\xi} z(s)\mathrm{d}s\right)\chi_2(h(t)),$$

where

$$\chi_2(s) = \begin{cases} 1 & \text{if } s \geq t_2, \\ 0 & \text{if } s < t_2. \end{cases}$$

After substituting (14) and (15) into (13), we have ($t \geq t_2$)

$$z(t) - u(t)\int_{t_2}^{t} \mathrm{e}^{-\int_s^t u(\xi)\mathrm{d}\xi} z(s)\mathrm{d}s + \int_{t_2}^{t} \mathrm{e}^{-\int_s^t a(\xi)\mathrm{d}\xi} b(s)\chi_2(h(s)) \int_{t_2}^{\max\{t_2,h(s)\}} \mathrm{e}^{-\int_\tau^{\max\{t_2,h(s)\}} u(\xi)\mathrm{d}\xi} z(\tau)\mathrm{d}\tau\mathrm{d}s$$
$$= f(t).$$

Hence

$$\begin{aligned} z(t) &- u(t)\int_{t_2}^{t} \mathrm{e}^{-\int_s^t u(\xi)\mathrm{d}\xi} z(s)\mathrm{d}s + \int_{t_2}^{t} \mathrm{e}^{-\int_s^t a(\xi)\mathrm{d}\xi} b(s)\chi_2(h(s)) \int_{t_2}^{s} \mathrm{e}^{-\int_\tau^{\max\{t_2,h(s)\}} u(\xi)\mathrm{d}\xi} z(\tau)\mathrm{d}\tau\mathrm{d}s \\ &- \int_{t_2}^{t} \mathrm{e}^{-\int_s^t a(\xi)\mathrm{d}\xi} b(s)\chi_2(h(s)) \int_{\max\{t_2,h(s)\}}^{s} \mathrm{e}^{-\int_\tau^{\max\{t_2,h(s)\}} u(\xi)\mathrm{d}\xi} z(\tau)\mathrm{d}\tau\mathrm{d}s \\ &= f(t). \end{aligned} \tag{16}$$

Since

$$\begin{aligned} &\int_{t_2}^{t}\left[\mathrm{e}^{-\int_s^t a(\xi)\mathrm{d}\xi} b(s)\chi_2(h(s)) \int_{t_2}^{s} \mathrm{e}^{-\int_\tau^{\max\{t_2,h(s)\}} u(\xi)\mathrm{d}\xi} z(\tau)\mathrm{d}\tau\right]\mathrm{d}s \\ &= \int_{t_2}^{t}\left[\int_{\tau}^{t} \mathrm{e}^{-\int_s^t a(\xi)\mathrm{d}\xi} b(s)\chi_2(h(s))\mathrm{e}^{-\int_\tau^{\max\{t_2,h(s)\}} u(\xi)\mathrm{d}\xi}\mathrm{d}s\right] z(\tau)\mathrm{d}\tau \\ &= \int_{t_2}^{t} \mathrm{e}^{-\int_\tau^t u(\xi)\mathrm{d}\xi}\left[\int_{\tau}^{t} \mathrm{e}^{-\int_s^t a(\xi)\mathrm{d}\xi} \mathrm{e}^{\int_{\max\{t_2,h(s)\}}^t u(\xi)\mathrm{d}\xi} b(s)\chi_2(h(s))\mathrm{d}s\right] z(\tau)\mathrm{d}\tau, \end{aligned}$$

Eq. (16) can be rewritten as

$$\begin{aligned} z(t) &- \int_{t_2}^{t} \mathrm{e}^{-\int_\tau^t u(\xi)\mathrm{d}\xi}\left[u(t) - \int_{\tau}^{t} \mathrm{e}^{-\int_s^t a(\xi)\mathrm{d}\xi} \mathrm{e}^{\int_{\max\{t_2,h(s)\}}^t u(\xi)\mathrm{d}\xi} b(s)\chi_2(h(s))\mathrm{d}s\right] z(\tau)\mathrm{d}\tau \\ &- \int_{t_2}^{t} \mathrm{e}^{-\int_s^t a(\xi)\mathrm{d}\xi} b(s)\chi_2(h(s)) \int_{\max\{t_2,h(s)\}}^{s} \mathrm{e}^{-\int_\tau^{\max\{t_2,h(s)\}} u(\xi)\mathrm{d}\xi} z(\tau)\mathrm{d}\tau\mathrm{d}s \\ &= f(t). \end{aligned} \tag{17}$$

On every finite interval $[t_2, T]$, $t_2 < T$, Eq. (17) has the operator form

$$z - Hz = f, \quad t \in [t_2, T]. \tag{18}$$

The operator $H: L[t_2, T] \to L[t_2, T]$ is bounded. In order to show that this operator is compact, we will apply Lemma 3. H can be rewritten in the form $H = PH_1 - H_2$ where

$$(Pz)(t) = u(t)z(t), \qquad (H_1 z)(t) = \int_{t_2}^{t} \mathrm{e}^{-\int_\tau^t u(\xi)\mathrm{d}\xi} z(\tau)\mathrm{d}\tau,$$

$$(H_2z)(t) = \int_{t_2}^{t}\left[e^{-\int_\tau^t u(\xi)d\xi}\int_\tau^t e^{-\int_s^t a(\xi)d\xi}e^{\int_{\max\{t_2,h(s)\}}^t u(\xi)d\xi}b(s)\chi_2(h(s))ds\right]z(\tau)d\tau$$
$$-\int_{t_2}^{t} e^{-\int_s^t a(\xi)d\xi}b(s)\chi_2(h(s))\int_{\max\{t_2,h(s)\}}^{s} e^{-\int_\tau^{\max\{t_2,h(s)\}} u(\xi)d\xi}z(\tau)d\tau ds.$$

It is easy to see that, for operators H_1, H_2, all conditions of Lemma 3 hold. Then these operators are compact. Moreover, the operator P is bounded, which means that H is a compact Volterra integral operator with spectral radius $r(T) = 0$ [21]. Hence, for the solution of Eq. (18), we have $z = (I - H)^{-1}f$ where I is the identity operator.

Operator H is (due to (12), (17) and $t_2 \le \tau$) a positive operator. It means that inequality $z(t) \ge 0$ implies $(Hz)(t) \ge 0$. Hence $(I - H)^{-1} = I + H + H^2 + H^3 + \cdots$ is also a positive operator.

Suppose now that, in Eq. (18), we have $f(t) \ge 0$. Then, for the solution of (18), we have $z(t) \ge 0$. The equality (14) implies that, for every right-hand side $f(t) \ge 0$, the solution of Eq. (13) $y(t)$ is non-negative. But, from (10), we get $y(t) = \int_{t_2}^t Y(t,s)f(s)ds$. Hence $Y(t,s) \ge 0, t_2 \le s < t \le T$. Since T can be chosen arbitrarily large, we conclude: $Y(t,s) \ge 0$, $t_2 \le s < t < \infty$.

We only have to prove that the strong inequality for $Y(t,s) > 0$ holds. After substituting

$$y(t) = \begin{cases} e^{-\int_{t_2}^t u(\xi)d\xi} & \text{if } t \ge t_2,\\ 0 & \text{if } t < t_2\end{cases}$$

into the left-hand side of Eq. (13), we see that this function is (due to (12)) a solution of (13) with a function $f(t) \le 0$ and with initial conditions

$$y(t) = 0, \quad t < t_2, \qquad y(t_2) = 1.$$

By the solution representation formula (10) with t_2 instead of t_0, we have

$$y(t) = Y(t,t_2) + \int_{t_2}^t Y(t,s)f(s)ds.$$

Hence $Y(t,t_2) \ge y(t) > 0$. The general case $Y(t,s) > 0$ is considered in much the same way. To finish the proof of this part, we set $t_3 := t_2$.

Implications (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (1) are evident. □

Remark 1. The proof of part (3) of Theorem 1 was finished by setting $t_3 := t_2$. Thus, it is easy to see that, if condition (2) of Theorem 1 holds for $t \ge t_0$ (instead of $t \ge t_2$), then $Y(t,s) > 0$, for $t > s \ge t_0$ (instead of $t > s \ge t_3$). This property is used below.

4. Positive solutions

The following lemma provides a link between Eqs. (2) and (7).

Lemma 4. *Let $x_1(t)$, $x_2(t)$, and $X(t,s)$ be the fundamental system and the fundamental function of (2), respectively, and let $Y(t,s)$ be the fundamental function of (7). Then*

$$x_1(t) = Y(t,t_0), \qquad x_2(t) = \int_{t_0}^t Y(t,\tau)e^{-\int_{t_0}^\tau a(\xi)d\xi}d\tau, \tag{19}$$

$$X(t,s) = \int_s^t Y(t,\tau)e^{-\int_s^\tau a(\xi)d\xi}d\tau. \tag{20}$$

Proof. Let $x: \mathbb{R} \to \mathbb{R}$ be a solution of Eq. (2) with a zero initial function for $t < t_0$. Rewriting it in the form

$$\left(\dot{x}(t)e^{\int_{t_0}^t a(\xi)d\xi}\right)' + b(t)e^{\int_{t_0}^t a(s)ds}x(h(s)) = 0$$

and integrating over the interval $[t_0, t]$, we have

$$\dot{x}(t) = e^{-\int_{t_0}^t a(\xi)d\xi}\dot{x}(t_0) - \int_{t_0}^t e^{-\int_s^t a(\xi)d\xi}b(s)x(h(s))ds, \quad t \ge t_0.$$

Hence

$$\dot{x}(t) + \int_{t_0}^t e^{-\int_s^t a(\xi)d\xi}b(s)x(h(s))ds = e^{-\int_{t_0}^t a(\xi)d\xi}\dot{x}(t_0)$$

and (by Lemma 2)

$$x(t) = Y(t, t_0)x(t_0) + \left[\int_{t_0}^{t} Y(t, \tau)\mathrm{e}^{-\int_{t_0}^{\tau} a(\xi)\mathrm{d}\xi}\mathrm{d}\tau\right]\dot{x}(t_0).$$

But for the solution of (2), we have a different representation (see Lemma 1):

$$x(t) = x_1(t)x(t_0) + x_2(t)\dot{x}(t_0).$$

Comparing both representations, we see that equalities (19) for the fundamental system of (2) are proved. Since $x_2(t) = X(t, t_0)$, the proof of equality (20) for $X(t, s)$ is similar. □

Corollary 1. *The fundamental function $Y(t, s)$ of (7) is positive for $t > s \geq t_0$ if and only if the fundamental system $x_1(t), x_2(t)$ and the fundamental function $X(t, s)$ of (2) are positive for $t > s \geq t_0$.*

In addition to (2) and (7), consider the following equations

$$\ddot{x}(t) + a_1(t)\dot{x}(t) + b_1(t)x(h_1(t)) = 0, \quad t \geq t_0 \tag{21}$$

and

$$\dot{y}(t) + \int_{t_0}^{t} \mathrm{e}^{-\int_{s}^{t} a_1(\xi)\mathrm{d}\xi} b_1(s)y(h_1(s))\mathrm{d}s = 0, \quad t \geq t_0 \tag{22}$$

where, for a_1, b_1, h_1, conditions relevant to (a1), (a2) hold. Denote $X_1(t, s)$ the fundamental function of (21) and $Y_1(t, s)$ the fundamental function of (22).

Corollary 2. *Let $a_1(t) \geq a(t) \geq 0$, $b(t) \geq b_1(t) \geq 0$, $h_1(t) \geq h(t)$, $t \geq t_0$ and let the fundamental system $x_1(t), x_2(t)$ and the fundamental function $X(t, s)$ of (2) be positive for $t > s \geq t_0$. Then, for some $t_1 \geq t_0$, the fundamental system and the fundamental function of (21) are positive for $t > s \geq t_1$.*

Proof. By Corollary 1, the fundamental function of Eq. (7) is positive. Hence, by Theorem 1, for some $t_1 \geq t_0$, inequality (12) (with t_1 instead of t_2) has a solution $u(t) \geq 0$, $t \geq t_1$. This function also is a nonnegative solution of inequality (12) where $a(t)$, $b(t)$ and $h(t)$ are replaced by $a_1(t)$, $b_1(t)$ and $h_1(t)$. Theorem 1, applied to (22) instead of (7), proves the positiveness of the fundamental function of (22). Finally, Corollary 1 (applied to (21) and (22) instead of (2) and (7)) implies this corollary. □

The following theorem improves Corollary 2 for the case $h(t) = h_1(t)$ since the condition $b_1(t) \geq 0$ is not assumed.

Theorem 2. *Let $a_1(t) \geq a(t) \geq 0$, $b(t) \geq b_1(t)$, $b(t) \geq 0$, $h_1(t) = h(t)$, $t \geq t_0$. If the fundamental function and the fundamental system of (2) are positive for $t > s \geq t_0$, then the fundamental function and the fundamental system of (21) are positive for $t > s \geq t_0$.*

Proof. By Corollary 1, the fundamental function of Eq. (7) is positive. Denote by $Y_1(t, s)$ the fundamental function of (22). By definition, the function $Y_1(t, t_0)$ is the solution of the problem

$$\dot{y}(t) + \int_{t_0}^{t} \mathrm{e}^{-\int_{s}^{t} a_1(\xi)\mathrm{d}\xi} b_1(s)y(h(s))\mathrm{d}s = 0, \quad t \geq t_0, \tag{23}$$
$$y(t) = 0, \quad t < t_0, \qquad y(t_0) = 1,$$

which can be rewritten in the form

$$\dot{y}(t) + \int_{t_0}^{t} \mathrm{e}^{-\int_{s}^{t} a(\xi)\mathrm{d}\xi} b(s)y(h(s))\mathrm{d}s = r(t), \quad t \geq t_0,$$
$$y(t) = 0, \quad t < t_0, \qquad y(t_0) = 1$$

where

$$r(t) = \int_{t_0}^{t} R(t, s)y(h(s))\mathrm{d}s$$

and

$$R(t, s) = \left(\mathrm{e}^{-\int_{s}^{t} a(\xi)\mathrm{d}\xi} - \mathrm{e}^{-\int_{s}^{t} a_1(\xi)\mathrm{d}\xi}\right) b(s) + \mathrm{e}^{-\int_{s}^{t} a_1(\xi)\mathrm{d}\xi}(b(s) - b_1(s)).$$

Then, for solution of (23), we have (by Lemma 2)

$$y(t) = g(t) + \int_{t_0}^{t} Y(t, s)\int_{t_0}^{s} R(s, \tau)y(h(\tau))\mathrm{d}\tau\mathrm{d}s,$$

where $g(t) = Y(t, t_0)$. Denote, for a fixed $T > t_0$,

$$A = \sup_{t_0 \le s \le t \le T} |R(t, s)|, \qquad B = \sup_{t_0 \le s \le t \le T} |Y(t, s)|,$$

and consider a linear operator

$$(Ty)(t) = \begin{cases} \int_{t_0}^{t} Y(t, s) \int_{t_0}^{s} R(s, \tau) y(h(\tau)) \mathrm{d}\tau \mathrm{d}s, & t \ge t_0, \\ y(t) = 0, \quad t < t_0 \end{cases}$$

in the space of continuous functions $C[t_0, T]$ with the norm

$$\|y\|_\lambda = \sup_{t_0 \le t \le T} \mathrm{e}^{-\lambda t} |y(t)|, \quad \lambda > 0.$$

We have

$$\begin{aligned} \|Ty\|_\lambda &= \sup_{t_0 \le t \le T} \left| \mathrm{e}^{-\lambda t} \int_{t_0}^{t} Y(t, s) \int_{t_0}^{s} R(s, \tau) y(h(\tau)) \mathrm{d}\tau \mathrm{d}s \right| \\ &\le AB \sup_{t_0 \le t \le T} \mathrm{e}^{-\lambda t} \int_{t_0}^{t} \int_{t_0}^{s} \mathrm{e}^{\lambda h(\tau)} \mathrm{e}^{-\lambda h(\tau)} |y(h(\tau))| \mathrm{d}\tau \mathrm{d}s \\ &\le AB \|y\|_\lambda \sup_{t_0 \le t \le T} \mathrm{e}^{-\lambda t} \int_{t_0}^{t} \int_{t_0}^{s} \mathrm{e}^{\lambda \tau} \mathrm{d}\tau \mathrm{d}s \le \frac{AB}{\lambda^2} \|y\|_\lambda. \end{aligned}$$

We choose $\lambda > 0$ such that $AB < \lambda^2$. Then $\|T\|_\lambda < 1$. Hence,

$$y(t) = Y_1(t, t_0) = g(t) + (Tg)(t) + (T^2 g)(t) + \cdots \ge g(t) = Y(t, t_0) > 0.$$

Similarly, $Y_1(t, s) \ge Y(t, s) > 0,\ t > s$.

Corollary 1 (applied to (21) and (22) instead of (2) and (7)) implies the statement of the theorem. □

In the following the symbol w^+, where w is a function, is defined as $w^+ = \max\{w, 0\}$.

Corollary 3. *Let $a(t) \ge 0$ and let the fundamental function and the fundamental system of the equation*

$$\ddot{x}(t) + a(t)\dot{x}(t) + b^+(t) x(h(t)) = 0 \tag{24}$$

be positive for $t > s \ge t_0$. Then the fundamental function and the fundamental system of Eq. (2) are positive for $t > s \ge t_0$.

Proof. In Theorem 2, as an equation of type (2), we take Eq. (24), and, as an equation of type (21), we take Eq. (2).
Since $b^+(t) \ge b(t)$, by Theorem 2, the fundamental function and the fundamental system of Eq. (2) are positive. □

Now we can formulate the most general comparison theorem.

Theorem 3. *Let $a_1(t) \ge a(t) \ge 0$, $b^+(t) \ge b_1^+(t)$, $h_1(t) \ge h(t),\ t \ge t_0$. If there exists a locally essentially bounded function u: $[t_0, \infty) \to \mathbb{R}$, $u(t) \ge 0$, $t \ge t_0$ such that*

$$u(t) \ge \int_{t_0}^{t} \mathrm{e}^{-\int_s^t a(\xi)\mathrm{d}\xi} \mathrm{e}^{\int_{\max\{t_0, h(s)\}}^{t} u(\xi)\mathrm{d}\xi} b^+(s) \mathrm{d}s, \quad t \ge t_0,$$

then the fundamental function and the fundamental system of (2) are positive for $t > s \ge t_0$ and, for some $t_1 \ge t_0$, the fundamental function and the fundamental system of (21) are positive for $t > s \ge t_1$.

Proof. By Theorem 1 and Corollary 1, the fundamental function and the fundamental system of (24) are positive for $t > s \ge t_0$. By Corollary 3, the fundamental function and the fundamental system of (2) are positive for $t > s \ge t_0$.
Compare now Eq. (24) with the following equation

$$\ddot{x}(t) + a_1(t)\dot{x}(t) + b_1^+(t) x(h_1(t)) = 0. \tag{25}$$

Corollary 2 implies that, for some $t_1 \ge t_0$, the fundamental function and the fundamental system of (25) are positive for $t > s \ge t_1$. By Corollary 3 (applied to (25)), the fundamental function and the fundamental system of (21) are also positive for $t > s \ge t_1$. □

Now let us proceed to explicit sufficient conditions when Eq. (2) has a positive solution. In the following theorem and its corollaries, denote by $\|\cdot\|$ the norm in the space $L_\infty[t_0, \infty)$.

Theorem 4. *If, for $t \ge t_0$, $t - h(t) \le \delta$, $\delta > 0$, there exists a locally essentially bounded nonnegative function $u(t)$, $t \ge t_0$ and a positive constant a_0 such that*

$$a(t) - u(t) \ge a_0, \qquad u(t) \ge \mathrm{e}^{\|u(t)\|\delta} \cdot \left\| \frac{b^+(t)}{a(t) - u(t)} \right\|,$$

then the fundamental function and the fundamental system of Eq. (2) are positive for $t > s \ge t_0$.

Proof. We will show that the function u is a solution of inequality (12) where t_2 is replaced by t_0. In view of Corollary 3, it is sufficient to prove the theorem for the case of $b(t) \ge 0$. Rewrite inequality (12):

$$u(t) \ge \int_{t_0}^{t} \mathrm{e}^{-\int_s^t (a(\xi)-u(\xi))\mathrm{d}\xi} \mathrm{e}^{\int_{\max\{t_0,h(s)\}}^{s} u(\xi)\mathrm{d}\xi} b(s)\mathrm{d}s, \quad t \ge t_0.$$

We have

$$\mathrm{e}^{\int_{\max\{t_0,h(s)\}}^{s} u(\xi)\mathrm{d}\xi} \le \mathrm{e}^{\delta\|u(t)\|}.$$

For every nonnegative integrable function $c(t) \ge 0$, $t \ge t_0$, we have

$$\begin{aligned} \int_{t_0}^{t} \mathrm{e}^{-\int_s^t c(\xi)\mathrm{d}\xi} c(s)\mathrm{d}s &= \int_{t_0}^{t} \mathrm{e}^{\int_t^s c(\xi)\mathrm{d}\xi} \left(\int_t^s c(\xi)\mathrm{d}\xi \right)'_s \mathrm{d}s \\ &= \mathrm{e}^{\int_t^s c(\xi)\mathrm{d}\xi} \Big|_{t_0}^{t} = 1 - \mathrm{e}^{-\int_{t_0}^{t} c(\xi)\mathrm{d}\xi} \le 1. \end{aligned}$$

Then

$$\begin{aligned} \int_{t_0}^{t} \mathrm{e}^{-\int_s^t (a(\xi)-u(\xi))\mathrm{d}\xi} b(s)\mathrm{d}s &= \int_{t_0}^{t} \mathrm{e}^{-\int_s^t (a(\xi)-u(\xi))\mathrm{d}\xi} (a(s) - u(s)) \frac{b(s)}{a(s) - u(s)} \mathrm{d}s \\ &\le \left\| \frac{b(t)}{a(t) - u(t)} \right\|. \end{aligned}$$

Thus (12) holds. Now it remains to apply Theorem 1, Remark 1 and Corollary 1. □

Corollary 4. *Let $t - h(t) \le \delta$, $\delta > 0$, $t \ge t_0$, and let there exist positive numbers λ, a_0 such that at least one of the following conditions holds*

(1) $a(t) - \lambda \ge a_0,\ \lambda \ge \mathrm{e}^{\lambda\delta} \cdot \left\| \frac{b^+(t)}{a(t)-\lambda} \right\|$,

(2) $\lambda < 1,\ a(t) \ge a_0/(1-\lambda),\ \lambda(1-\lambda)a(t) \ge \mathrm{e}^{\lambda\|a(t)\|\delta} \cdot \left\| \frac{b^+(t)}{a(t)} \right\|$.

Then the fundamental function and the fundamental system of (2) are positive for $t > s \ge t_0$.

Proof. To prove (1), we use $u(t) = \lambda$ in Theorem 4; to prove (2), we use $u(t) = \lambda a(t)$ in Theorem 4. □

Corollary 5. *Let $t - h(t) \le \delta$, $\delta > 0$, $t \ge t_0$, and let there exist a positive number a_0 such that at least one of the following conditions holds*

(1) $a(t) - \frac{1}{\delta} \ge a_0,\ \left\| \frac{b^+(t)}{a(t)-\delta^{-1}} \right\| \le \frac{1}{\delta e}$,

(2) $a(t) \equiv 2a_0,\ (a_0)^2 \ge \|b^+(t)\|\mathrm{e}^{a_0\delta/2}$.

Then the fundamental function and the fundamental system of (2) are positive for $t > s \ge t_0$.

Proof. To prove (1), we use $\lambda = 1/\delta$ in Corollary 4 (case 1); to prove (2), we use $\lambda = 1/2$ in Corollary 4 (case 2). □

Now let us compare the fundamental systems with the fundamental functions of two different equations.

Theorem 5. *Let the fundamental system $x_1(t), x_2(t)$ and the fundamental function $X(t,s)$ of (2) be positive for $t > s \ge t_0$. Denote by $v_1(t)$, $v_2(t)$, $V(t,s)$ the fundamental system and the fundamental function of (2) where $b(t)$ is replaced by $b_1(t)$. If $b(t) \ge b_1(t) \ge 0$, $t \ge t_0$, then*

$$x_1(t) \le v_1(t), \qquad x_2(t) \le v_2(t), \qquad X(t,s) \le V(t,s), \quad t > s \ge t_0.$$

Proof. Denote by $Y(t,s)$ the fundamental function of (7) and by $Y_1(t,s)$ the fundamental function of (7) where $b(t)$ is replaced by $b_1(t)$. By Corollary 1 and Lemma 4, it is sufficient to prove that $Y_1(t,s) \ge Y(t,s)$.

The function $y(t) = Y_1(t, t_0)$ is the solution of the problem

$$\dot{y}(t) + \int_{t_0}^{t} e^{-\int_s^t a(\xi)d\xi} b_1(s)y(h(s))ds = 0,$$
$$y(t) = 0, \quad t < t_0, \qquad y(t_0) = 1,$$

which can be rewritten in the form

$$\dot{y}(t) + \int_{t_0}^{t} e^{-\int_s^t a(\xi)d\xi} b(s)y(h(s))ds = \int_{t_0}^{t} e^{-\int_s^t a(\xi)d\xi} (b(s) - b_1(s))y(h(s))ds, \tag{26}$$
$$y(t) = 0, \quad t < t_0, \qquad y(t_0) = 1, \tag{27}$$

By formula (10), for the solution of (26), (27), we have

$$y(t) = Y(t, t_0)y(t_0) + \int_{t_0}^{t} Y(t, s) \int_{t_0}^{s} e^{-\int_\tau^s a(\xi)d\xi} (b(\tau) - b_1(\tau))y(h(\tau))d\tau ds.$$

Since $b(t) \geq b_1(t) \geq 0, y(t) = Y_1(t, t_0), y(t_0) = 1$, we have $Y_1(t, t_0) \geq Y(t, t_0)$. The inequality $Y_1(t, s) \geq Y(t, s)$ is obtained in much the same way. □

In the last part of the paper, we will state several open problems.

Open problem 1.
Extend the results of the paper to Eq. (2) without the assumption that $a(t) \geq 0$.

Open problem 2.
Extend the results of the paper to Eq. (4) with two delay terms.

Open problem 3.
Assuming that the fundamental system and the fundamental function for Eq. (2) are positive for $t \geq 0$, find conditions for the solution of the initial value problem (5), (6) to be positive. Such conditions for delay differential equations of the first order are well known (see, e.g., [4]).

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